Fundamentals of Statistical And Thermal Physics

Fall, 2014

Reference

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- Mehran Kardar, *Statistical Physics of Particles*
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- 王竹溪，《统计物理学导论》
- 林宗涵，《热力学与统计物理学》

Chapter 1

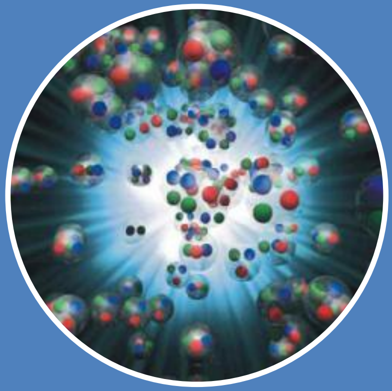
Introduction to statistical methods

Muhong Wu
9/18/2014

Contents

- Introduction
- Random Walk and Binomial Distribution
- General Discussion of the Random Walk

Introduction

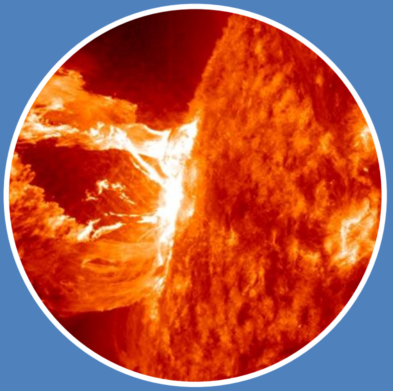


Microscopic

Classical mechanics
Quantum mechanics

Statistics

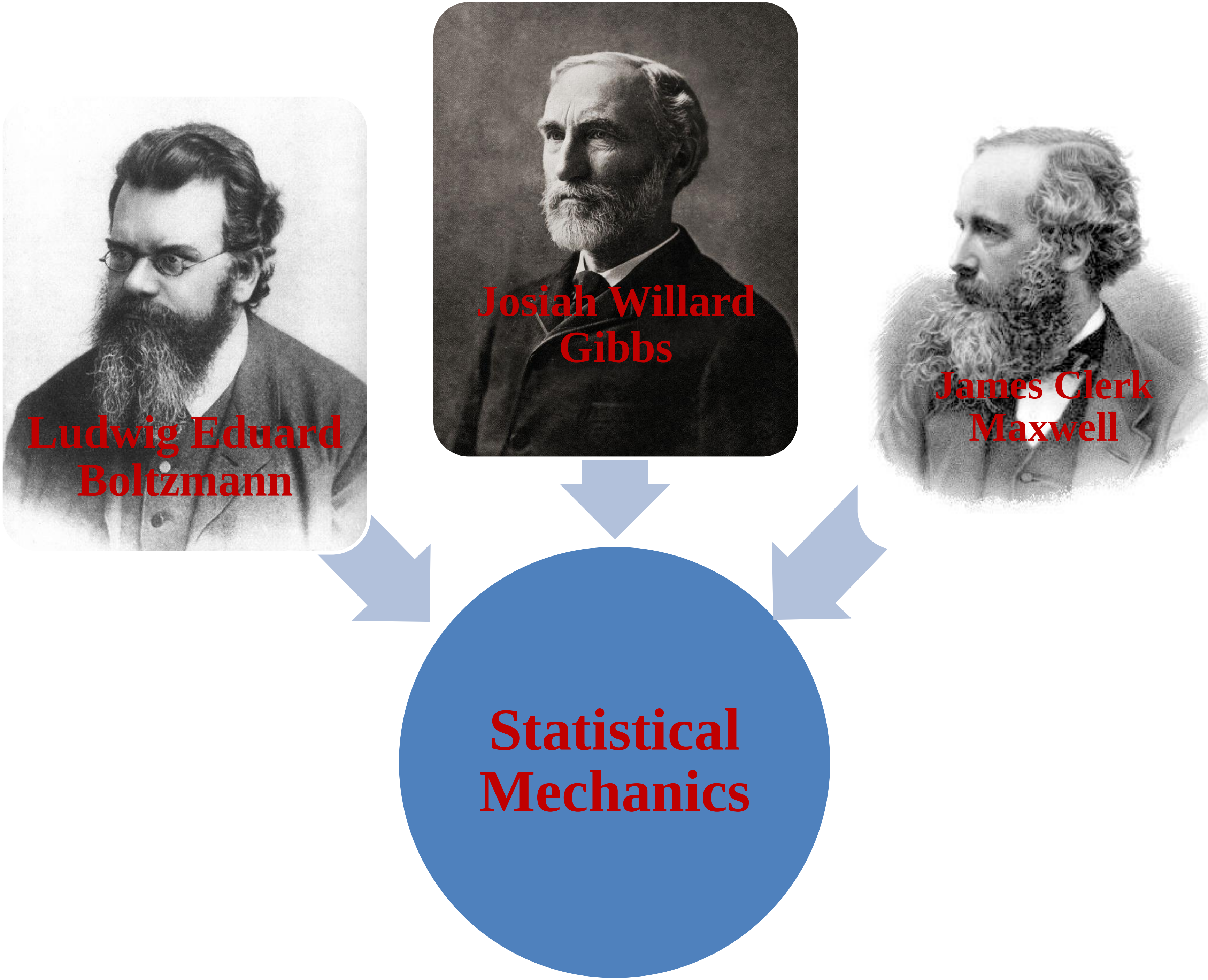
Statistical Physics/
Statistical
mechanics/
Statistical
thermodynamics



Macroscopic

Thermodynamics

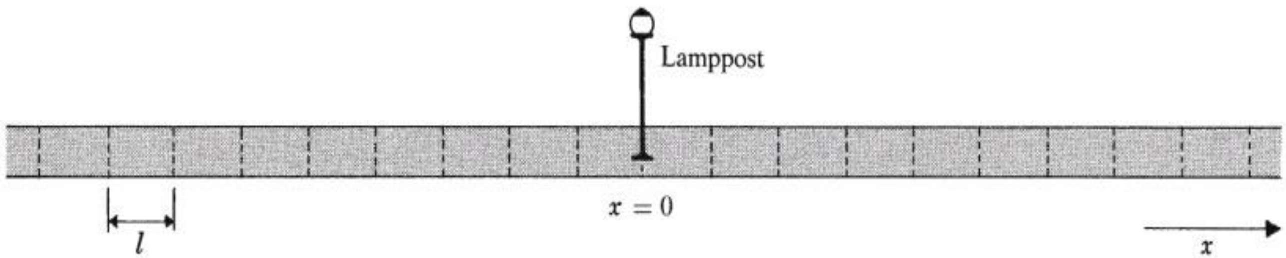
Connection


Ludwig Eduard Boltzmann
Josiah Willard Gibbs
James Clerk Maxwell
Statistical Mechanics

Random Walk and Binomial Distribution

- Elementary statistical concepts and examples
- The simple random walk problem in one dimension
- General discussion of mean values
- Calculation of mean values for random walk problem
- Probability distribution for large N
- Gaussian probability distribution

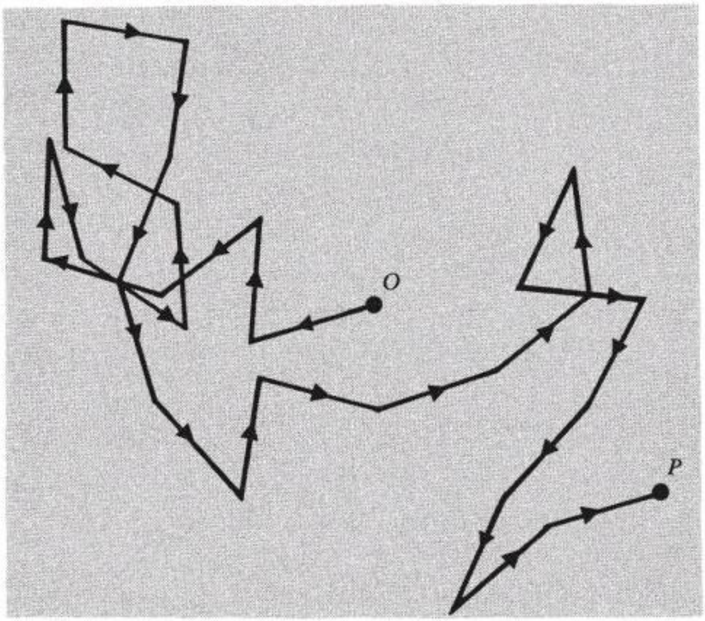
Elementary statistical concepts and examples



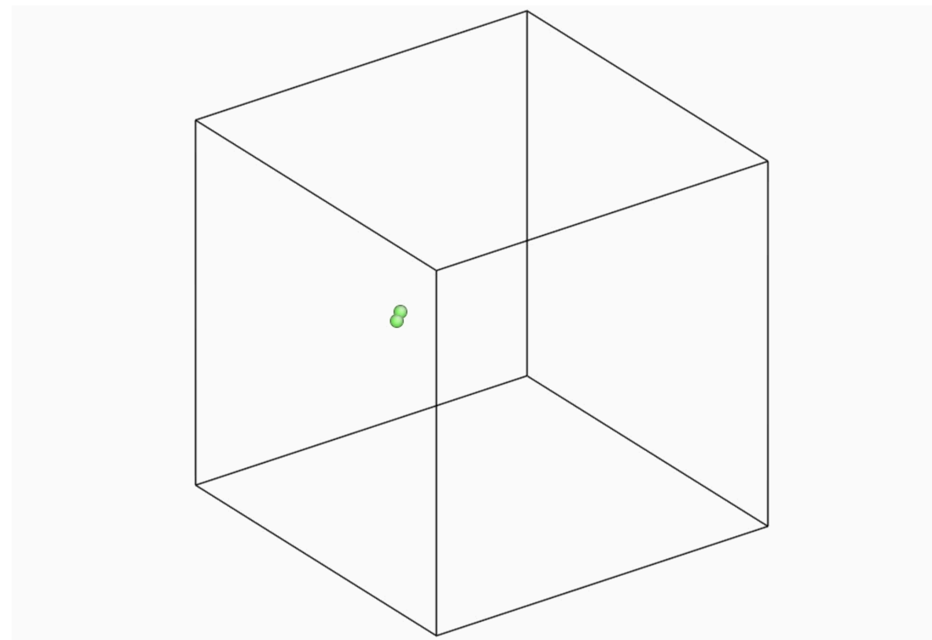
The drunkaed's random walk in one dimension

Each step the drunk takes is of equal length l. The probability to right is p, while the probability to left is $q = 1 - p$. After N steps, what's the probability $P_N(m)$ of his being located at the position $x = ml$?

Elementary statistical concepts and examples



Example of a random walk in two dimensions



Example of a random walk in three dimensions. The single self-interstitial migration in alpha-iron at 950K .

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The simple random walk problem in one dimension

$x = ml, \quad -\mathrm{N} \leq m \leq N$
$N = n_1 + n_2,$
$m = n_1 - n_2 = 2n_1 - N,$
$n_1 = \frac{1}{2}(N+m), n_2 = \frac{1}{2}(N-m)$

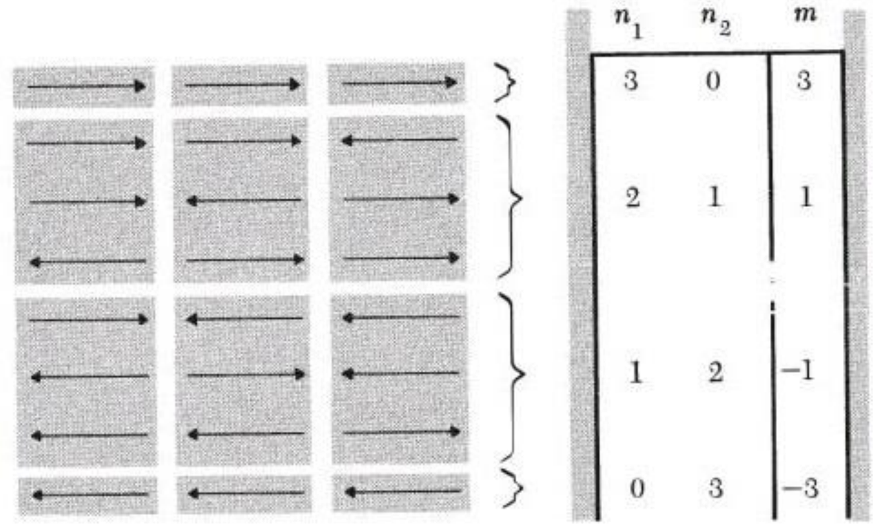
The probability $W_N(n_1)$ of taking n_1 steps to the right and n_2 steps to the left is

$$W_N(n_1) = \frac{N!}{n_1!\,n_2!} p^{n_1} q^{n_2}$$

The probability $P_N(m)$of his being located at the position $x = ml$ is

$$P_N(m) = W_N(n_1)$$

$$P_N(m) = \frac{N!}{[(N+m)/2]!\,[(N-m)/2]!} p^{(N+m)/2}(1-p)^{(N-m)/2}$$



The 8 sequences of steps which are possinble if the total number of steps $N = 3$.

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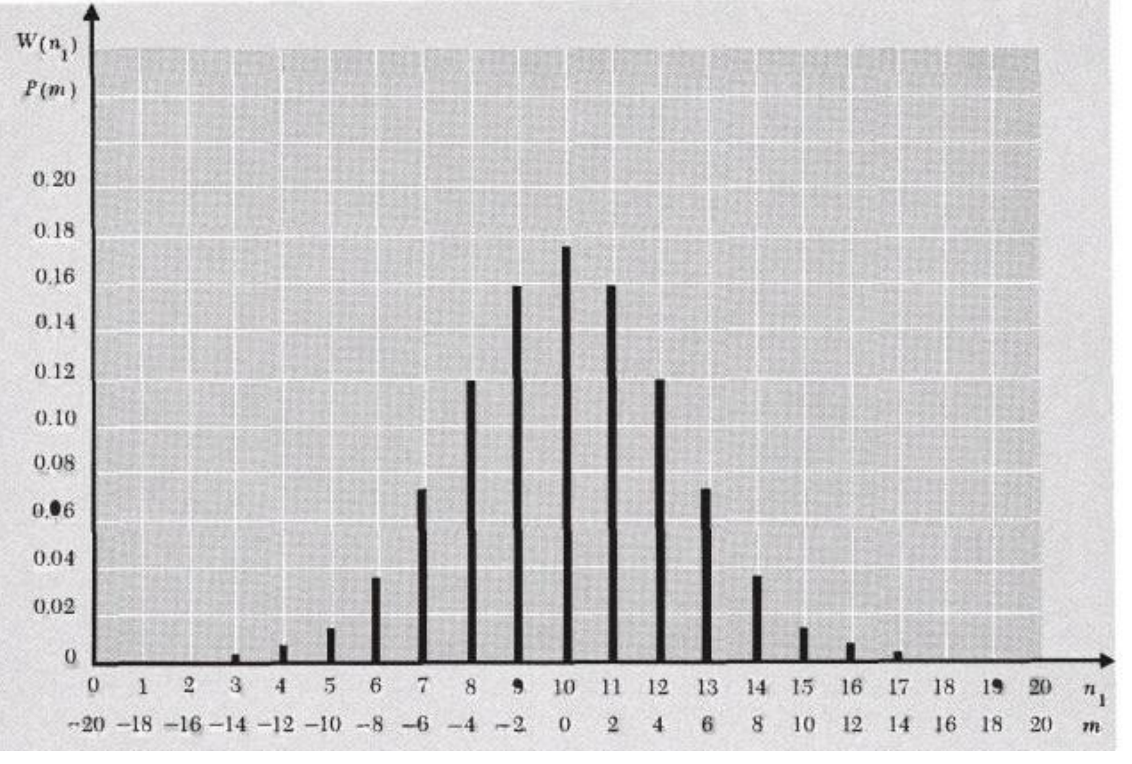
The simple random walk problem in one dimension

If $p = q = \frac{1}{2}$, $\quad P_N(m) = \dfrac{N!}{[(N+m)/2]!\,[(N-m)/2]!}\left(\frac{1}{2}\right)^N$



Binomial probability distribution for $p = q = \frac{1}{2}$ when $N = 20$ steps.

General discussion of mean values

The mean value of u is denoted by $\bar{u}$ and is defined by

$$\bar{u} \equiv \frac{\sum_{i=1}^{M} P(u_i) u_i}{\sum_{i=1}^{M} P(u_i)}$$

If the $f(u)$is any function of u, *the* mean value of $f(u)$ is defined by,

$$\overline{f(u)} \equiv \frac{\sum_{i=1}^{M} P(u_i) f(u_i)}{\sum_{i=1}^{M} P(u_i)}\,, \qquad \sum_{i=1}^{M} P(u_i) = 1$$

so,

$$\overline{f(u)} = \sum_{i=1}^{M} P(u_i) f(u_i)$$

and

$$\overline{f(u) + g(u)} = \overline{f(u)} + \overline{g(u)}$$

$$\overline{cf(u)} = \mathrm{c}\overline{f(u)}$$

$$\overline{\Delta u} = \overline{(u - \bar{u})} = \bar{u} - \bar{u} = 0, \quad \overline{\Delta u^2} = \overline{(u - \bar{u})^2} = \overline{u^2} - \bar{u}^2 \geq 0$$

Calculation of mean values for random walk problem

The mean number of n_1 of steps to the right is denoted by $\overline{n_1}$ and is defined by

$$\overline{n_1} \equiv \sum_{n_1=1}^{N} W(n_1)n_1 = \sum_{n_1=1}^{N} \frac{N!}{n_1!\,(N - n_1)!} p^{n_1} q^{N-n_1} n_1$$

and

$$\sum_{n_1=1}^{N} W(n_1) = 1$$

so

$$\overline{n_1} = Np$$

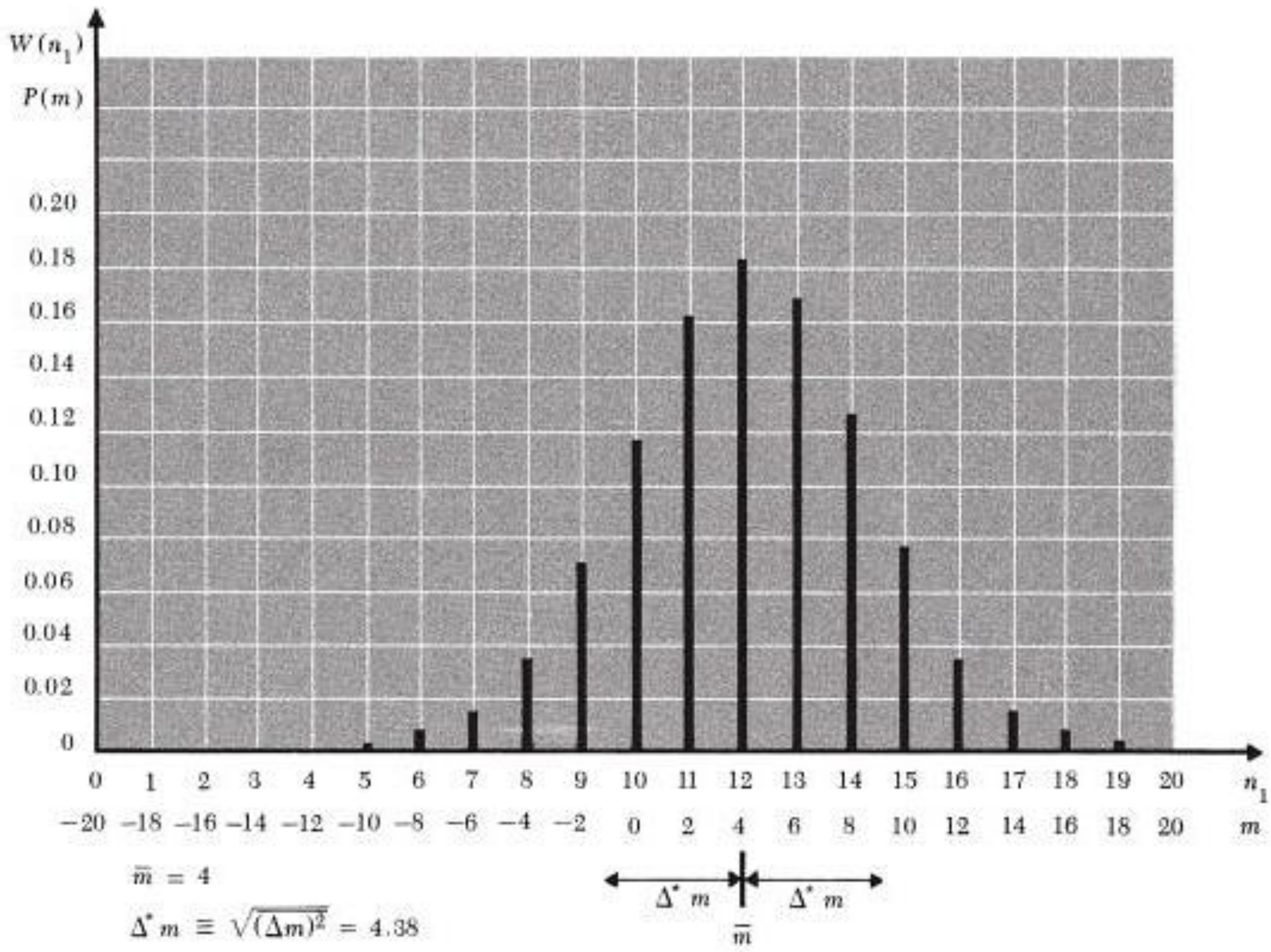
$$\bar{m} = \overline{n_1 - n_2} = \overline{n_1} - \overline{n_2} = N(p - q)$$

and

$$\overline{(\Delta n_1)^2} = Npq$$

$$\overline{(\Delta m)^2} = 4\overline{(\Delta n_1)^2} = 4Npq$$

Calculation of mean values for random walk problem



Binomial probability distribution for $p = 0.6$ $and\ q = 0.4$, when $N = 20$ steps.

Probability distribution for large N

The location $n_1 = \tilde{n}$ of the maximum of W is approximately determined by the condition

$$\frac{dW}{dn_1} = 0 \ or \ \frac{dlnW}{dn_1} = 0$$

where the derivatives are evaluated for $n_1 = \widetilde{n_1}$. To investigate the behavior of $W(n_1)$ near its maximum, we shall put

$$n_1 \equiv \widetilde{n_1} + \eta$$

and eapand $lnW(n_1)$ in a Taylor's series about $\widetilde{n_1}$.

$$\ln W(n_1) = \ln W(\tilde{n}_1) + B_1\eta + \tfrac{1}{2}B_2\eta^2 + \tfrac{1}{6}B_3\eta^3 + \cdots$$

where
$$B_k \equiv \frac{d^k \ln W}{dn_1{}^k}$$

In the region where η is sufficiently small, higher-order terms in the expansion can be neglected so that one obtains in first approximation an expression of the simple form

$$W(n_1) = \tilde{W} e^{-\frac{1}{2}|B_2|\eta^2}$$

Probability distribution for large N

so

$$n_1 = Np, and\ B_2 = -\frac{1}{Npq}$$

since $p + q = 1$. And

$$W(n_1) = \sqrt{\frac{|B_2|}{2\pi}}\, e^{-\frac{1}{2}|B_2|(n_1 - \tilde{n}_1)^2}$$

$$W(n_1) = (2\pi Npq)^{-\frac{1}{2}} \exp\left[-\frac{(n_1 - Np)^2}{2Npq}\right]$$

Gaussian probability distribution

The Gaussian approximation (1·5·19) also yields immediately the probability $P(m)$ that in a large number of N steps the net displacement is m. The corresponding number of right steps is, by (1·2·9), $n_1 = \frac{1}{2}(N + m)$. Hence (1·5·19) gives

$$P(m) = W\left(\frac{N+m}{2}\right) = [2\pi Npq]^{-\frac{1}{2}} \exp\left\{-\frac{[m - N(p-q)]^2}{8Npq}\right\} \qquad (1\cdot6\cdot1)$$

probability of finding the particle anywhere in the range between x and $x + dx$ is simply obtained by summing $P(m)$ over all values of m lying in dx, i.e., by multiplying $P(m)$ by $dx/2l$. This probability is thus proportional to dx (as one would expect) and can be written as

$$\mathcal{P}(x)\, dx = P(m)\, \frac{dx}{2l} \qquad (1\cdot6\cdot3)$$

where the quantity $\mathcal{P}(x)$, which is independent of the magnitude of dx, is called a "probability *density*." Note that it must be multiplied by a differential element of length dx to yield a probability.

By using (1·6·1) one then obtains

▶ $$\mathcal{P}(x)\, dx = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}\, dx \qquad (1\cdot6\cdot4)$$

where we have used the abbreviations

$$\mu \equiv (p - q)Nl \qquad (1\cdot6\cdot5)$$

and

$$\sigma \equiv 2\sqrt{Npq}\; l \qquad (1\cdot6\cdot6)$$

Gaussian probability distribution

And

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{P}(x)\,dx &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\sqrt{\pi 2\sigma^2} \\
&= 1
\end{aligned}
$$

The mean values

$$
\begin{aligned}
\bar{x} &\equiv \int_{-\infty}^{\infty} x\mathcal{P}(x)\,dx \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} x\, e^{-(x-\mu)^2/2\sigma^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\left[\int_{-\infty}^{\infty} y\, e^{-y^2/2\sigma^2}\,dy + \mu\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy\right]
\end{aligned}
$$

$$\bar{x} = \mu$$

$$
\begin{aligned}
\overline{(x-\mu)^2} &= \int_{-\infty}^{\infty} (x-\mu)^2\mathcal{P}(x)\,dx \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} y^2\, e^{-y^2/2\sigma^2}\,dy \\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\left[\frac{\sqrt{\pi}}{2}(2\sigma^2)^{\frac{3}{2}}\right] \\
&= \sigma^2
\end{aligned}
$$

$$\overline{(\Delta x)^2} = \overline{(x-\mu)^2} = \sigma^2$$

$$
\begin{aligned}
\bar{x} &= (p-q)Nl \\
\overline{(\Delta x)^2} &= 4Npql^2
\end{aligned}
$$

General Discussion of the Random Walk

- Probability distributions involving several variables
- Comments on continuous probability distribution
- General calculation of mean values for random walk
- Calculation of the probability distribution
- Probability distribution for large N

Probability distributions involving several variables

The statistical description of a situation involving more than one variable requires only straightforward generalizations of the probability arguments applicable to a single variable. Let us then, for simplicity, consider the case of only two variables u and v which can assume the possible values

u_i where $i = 1, 2, \ldots, M$

and v_j where $j = 1, 2, \ldots, N$

Let $P(u_i, v_j)$ be the probability that u assumes the value u_i *and* that v assumes the value v_j.

The probability that the variables u and v assume any of their possible sets of values must be unity; i.e., one has the normalization requirement

$$\sum_{i=1}^{M} \sum_{j=1}^{N} P(u_i, v_j) = 1 \qquad (1 \cdot 7 \cdot 1)$$

where the summation extends over all possible values of u and all possible values of v.

$$P_u(u_i) = \sum_{j=1}^{N} P(u_i, v_j)$$

$$P_v(v_j) = \sum_{i=1}^{M} P(u_i, v_j)$$

$$\sum_{i=1}^{M} P_u(u_i) = \sum_{i=1}^{M} \left[\sum_{j=1}^{N} P(u_i, v_j) \right] = 1$$

$$P(u_i, v_j) = P_u(u_i) P_v(v_j)$$

if u and v are statistically independent.

Let us now mention some properties of mean values. If $F(u,v)$ is any function of u and v, then its mean value is defined by

$$\overline{F(u,v)} \equiv \sum_{i=1}^{M} \sum_{j=1}^{N} P(u_i, v_j) F(u_i, v_j) \qquad (1 \cdot 7 \cdot 6)$$

Note that if $f(u)$ is a function of u only, it also follows by $(1 \cdot 7 \cdot 2)$ that

$$\overline{f(u)} = \sum_i \sum_j P(u_i, v_j) f(u_i) = \sum_i P_u(u_i) f(u_i) \qquad (1 \cdot 7 \cdot 7)$$

If F and G are any functions of u and v, then one has the general result

$$\begin{aligned} \overline{F + G} &\equiv \sum_i \sum_j P(u_i, v_j)[F(u_i, v_j) + G(u_i, v_j)] \\ &= \sum_i \sum_j P(u_i, v_j) F(u_i, v_j) + \sum_i \sum_j P(u_i, v_j) G(u_i, v_j) \end{aligned}$$

or

▶ $$\overline{F + G} = \overline{F} + \overline{G} \qquad (1 \cdot 7 \cdot 8)$$

▶ $$\overline{f(u) g(v)} = \overline{f(u)}\ \overline{g(v)} \qquad (1 \cdot 7 \cdot 9)$$

Comments on continuous probability distribution

One dimension

Fig. 1·8·1 Subdivision of the range $a_1 < u < a_2$ of a continuous variable u into a countable number of infinitesimal intervals δu of fixed size.

To make the connection between the continuous and discrete points of view quite explicit, note that in terms of the original infinitesimal subdivision interval δu,

$$P(u) = \mathcal{P}(u)\,\delta u$$

Similarly, if one considers any interval between u and $u + du$ which is such that du is macroscopically small although $du \gg \delta u$, then this interval contains $du/\delta u$ possible values of u_i for which the probability $P(u_i)$ has essentially the same value—call it simply $P(u)$. Then the probability $P(u)\,du$ of the variable assuming a value between u and $u + du$ should be given by multiplying the probability $P(u_i)$ for assuming any discrete value in this range by the number $du/\delta u$ of discrete values in this range; i.e., one has properly

$$\mathcal{P}(u)\,du = P(u_i)\,\frac{du}{\delta u} = \frac{P(u)}{\delta u}\,du \qquad (1\cdot8\cdot1)$$

$$\sum_i P(u_i) = 1 \qquad \int_{a_1}^{a_2} \mathcal{P}(u)\,du = 1$$

$$\overline{f(u)} = \sum_i P(u_i) f(u_i) \qquad \overline{f(u)} = \int_{a_1}^{a_2} \mathcal{P}(u) f(u)\,du$$

Two dimensions

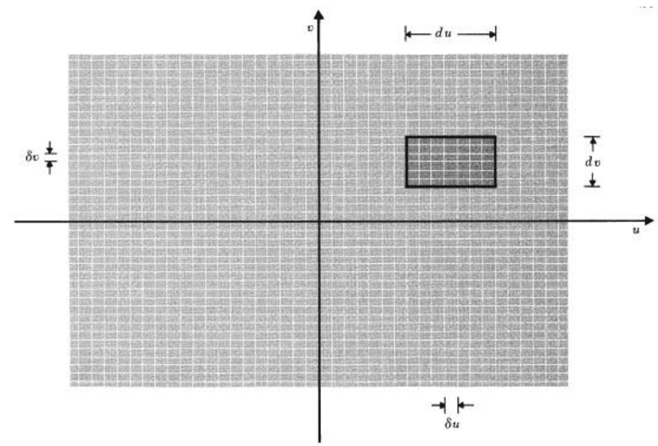


Fig. 1·8·2 Subdivision of the continuous variables u and v into small intervals of magnitude δu and δv.

$$\mathcal{P}(u,v)\,du\,dv = P(u,v)\,\frac{du\,dv}{\delta u\,\delta v}$$

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} du\,dv\,\mathcal{P}(u,v) = 1$$

$$\overline{F(u,v)} = \int_{a_1}^{a_2}\int_{b_1}^{b_2} du\,dv\,\mathcal{P}(u,v)F(u,v)$$

Comments on continuous probability distribution

Functions of random variables

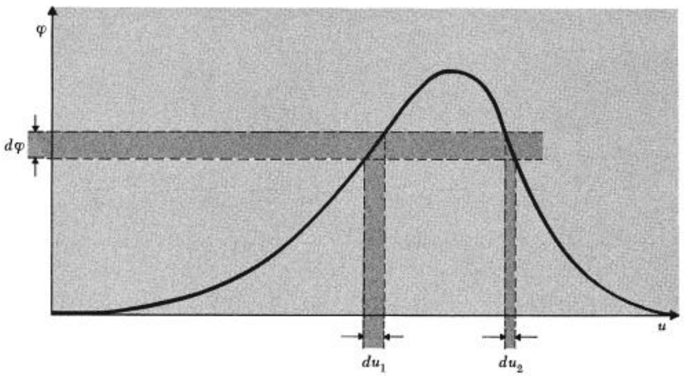


Fig. 1·8·3 Illustration showing a function $\varphi(u)$ which is such that $u(\varphi)$ is a double-valued function of φ. Here the range $d\varphi$ corresponds to u being either in the range du_1 or in the range du_2.

Functions of random variables Consider the case of a single variable u and suppose that $\varphi(u)$ is some continuous function of u. The following question arises quite frequently. If $\mathcal{P}(u)\,du$ is the probability that u lies in the range between u and $u + du$, what is the corresponding probability $W(\varphi)\,d\varphi$ that φ lies in the range between φ and $\varphi + d\varphi$? Clearly, the latter probability is obtained by adding up the probabilities for all those values u which are such that φ lies in the range between φ and $\varphi + d\varphi$; in symbols

$$W(\varphi)\,d\varphi = \int_{d\varphi} \mathcal{P}(u)\,du \qquad (1\cdot 8\cdot 8)$$

Here u can be considered a function of φ and the integral extends over all those values of u lying in the range between $u(\varphi)$ and $u(\varphi + d\varphi)$. Thus (1·8·8) becomes simply

$$W(\varphi)\,d\varphi = \int_{\varphi}^{\varphi+d\varphi} \mathcal{P}(u) \left|\frac{du}{d\varphi}\right| d\varphi = \mathcal{P}(u) \left|\frac{du}{d\varphi}\right| d\varphi \qquad (1\cdot 8\cdot 9)$$

The last step assumes that u is a single-valued function of φ and follows, since the integral is extended only over an infinitesimal range $d\varphi$. Since $u = u(\varphi)$, the right side of (1·8·9) can, of course, be expressed completely in terms of φ. If $u(\varphi)$ is not a single-valued function of φ, then the integral (1·8·8) may consist of several contributions similar to those of (1·8·9) (see Fig. 1·8·3).

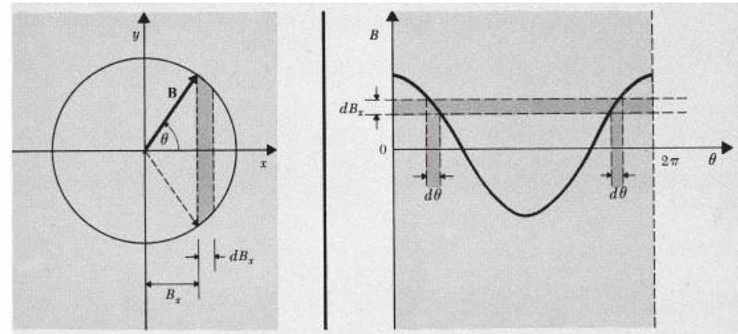


Fig. 1·8·4 Dependence of the x component $B_x = B\cos\theta$ of a two-dimensional vector B on its polar angle θ.

$$\mathcal{P}(\theta)\,d\theta = \frac{d\theta}{2\pi}$$

$$B_x = B\cos\theta$$

$$W(B_x)\,dB_x = 2\left[\frac{1}{2\pi}\frac{dB_x}{|B\sin\theta|}\right] = \frac{1}{\pi B}\frac{dB_x}{|\sin\theta|}$$

$$|\sin\theta| = (1-\cos^2\theta)^{\frac{1}{2}} = \left[1-\left(\frac{B_x}{B}\right)^2\right]^{\frac{1}{2}}$$

$$W(B_x)\,dB_x = \begin{cases} \dfrac{dB_x}{\pi\sqrt{B^2 - B_x^2}} & \text{for } -B \le B_x \le B \\ 0 & \text{otherwise} \end{cases}$$

General calculation of mean values for random walk

Functions of random variables

Let $w(s_i)\,ds_i$ be the probability that the ith displacement lies in the range between s_i and $s_i + ds_i$.

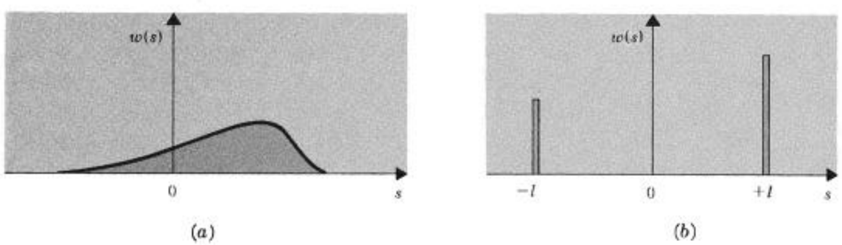


Fig. 1·9·1 Some examples of probability distributions giving, for any one step, the probability $w(s)\,ds$ that the displacement is between s and $s + ds$. (a) A rather general case, displacements to the right being more probable than those to the left. (b) The special case discussed in Sec. 1·2. Here the peaks, centered about $+l$ and $-l$, respectively, are very narrow; the area under the right peak is p, that under the left one is q. (The curves (a) and (b) are not drawn to the same scale; the total area under each should be unity.)

The total displacement x is equal to

$$x = s_1 + s_2 + \cdots + s_N = \sum_{i=1}^{N} s_i$$

Taking mean values of both sides,

$$\bar{x} = \overline{\sum_{i=1}^{N} s_i} = \sum_{i=1}^{N} \bar{s}_i$$

▶ $$\bar{x} = N\bar{s}$$

where $$\bar{s} \equiv \bar{s}_i = \int ds\, w(s)s$$

$$\overline{(\Delta x)^2} \equiv \overline{(x - \bar{x})^2}$$

$$x - \bar{x} = \sum_i (s_i - \bar{s})$$

or $$\Delta x = \sum_{i=1}^{N} \Delta s_i$$

where $$\Delta s = s_i - \bar{s}$$

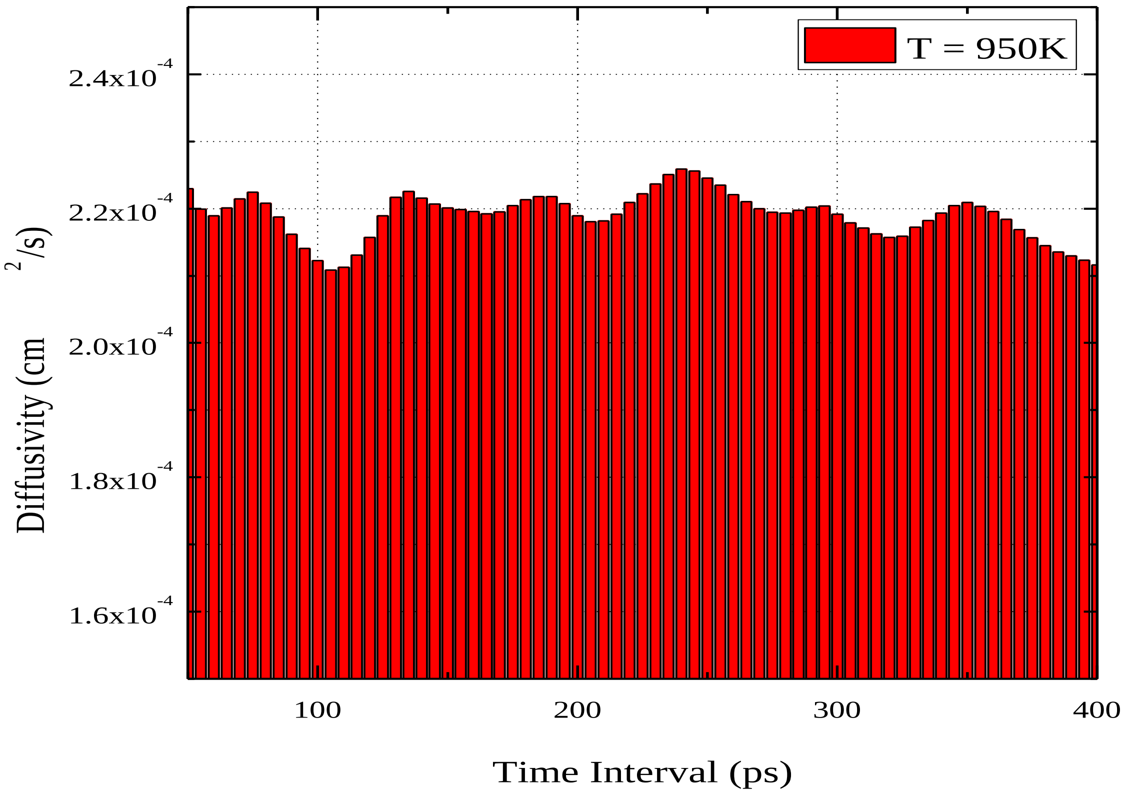
$$\overline{(\Delta x)^2} = \sum_{i=1}^{N} \overline{(\Delta s_i)^2}$$

▶ $$\overline{(\Delta x)^2} = N\overline{(\Delta s)^2}$$

where $$\overline{(\Delta s)^2} \equiv \overline{(\Delta s_i)^2} = \int ds\, w(s)(\Delta s)^2$$

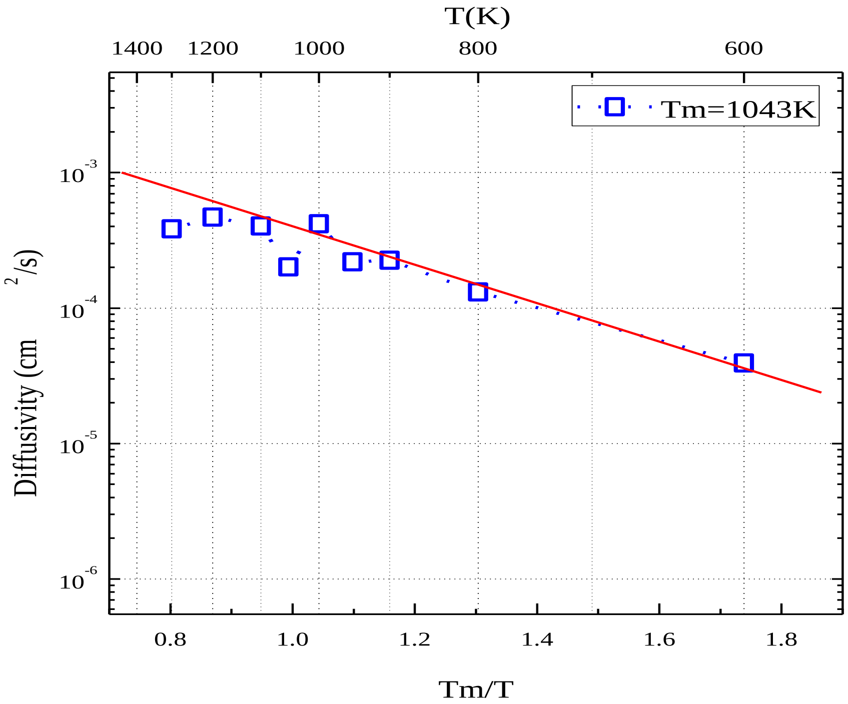
$$\frac{\Delta^* x}{\bar{x}} = \frac{\Delta^* s}{\bar{s}} \frac{1}{\sqrt{N}}$$

General calculation of mean values for random walk



The single self-interstitial diffusion in alpha-iron at 950K via SLD

General calculation of mean values for random walk



The single self-interstitial diffusion in alpha-iron at different tempreture via SLD

$$E \approx 2.63609 \times k_B T_m \approx 236.929 \text{ meV}$$

General calculation of mean values for random walk

For the problem discussed in the last section, the total displacement x in N steps is given by

$$x = \sum_{i=1}^{N} s_i \qquad (1\cdot 10\cdot 1)$$

We now want to find the probability $\mathcal{P}(x)\,dx$ of finding x in the range between x and $x + dx$. Since the steps are statistically independent, the probability of *a particular* sequence of steps where

the 1st displacement lies in the range between s_1 and $s_1 + ds_1$
the 2nd displacement lies in the range between s_2 and $s_2 + ds_2$
. . .
the Nth displacement lies in the range between s_N and $s_N + ds_N$

is simply given by the product of the respective probabilities, i.e., by

$$w(s_1)\,ds_1 \cdot w(s_2)\,ds_2 \cdots w(s_N)\,ds_N$$

If we sum this probability over all the possible individual displacements which are consistent with the condition that the total displacement x in $(1\cdot 10\cdot 1)$ always lies in the range between x and $x + dx$, then we obtain the total probability $\mathcal{P}(x)\,dx$, irrespective of the sequence of steps producing this total displacement. In symbols we can write

$$\mathcal{P}(x)\,dx = \underset{(dx)}{\int\!\!\int_{-\infty}^{\infty}\!\!\cdots\!\int} w(s_1)w(s_2)\cdots w(s_N)\,ds_1\,ds_2\cdots ds_N \qquad (1\cdot 10\cdot 2)$$

where the integration is over all possible values of the variables s_i, subject to the restriction that

$$x < \sum_{i=1}^{N} s_i < x + dx \qquad (1\cdot 10\cdot 3)$$

$$\mathcal{P}(x)\,dx = \int\!\!\int_{-\infty}^{\infty}\!\!\cdots\!\int w(s_1)w(s_2)\cdots w(s_N)\left[\delta\left(x - \sum_{i=1}^{N} s_i\right)dx\right]ds_1\,ds_2\cdots ds_N$$

$$\delta(x - \Sigma s_i) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik[\Sigma s_i - x]}$$

$$\mathcal{P}(x) = \int\!\!\int\cdots\int w(s_1)w(s_2)\cdots w(s_N)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(s_1+\cdots+s_N-x)}\,ds_1\,ds_2\cdots ds_N$$

or $$\mathcal{P}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}\int_{-\infty}^{\infty} ds_1\, w(s_1)\, e^{iks_1}\cdots\int_{-\infty}^{\infty} ds_N\, w(s_N)\, e^{iks_N}$$

▶ $$Q(k) \equiv \int_{-\infty}^{\infty} ds\, e^{iks} w(s)$$

Hence $(1\cdot 10\cdot 6)$ becomes

▶ $$\mathcal{P}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx} Q^N(k)$$

Probability distribution for large N

Remark To the extent that $w(s)$ varies slowly over a period of oscillation, the integral $Q(k) = \int ds\, e^{iks}w(s) \approx 0$. The reason is that in any range $a < s < b$ in which w varies slowly so that $|dw/ds|(b-a) \ll w$, but which contains many oscillations so that $(b-a)k \gg 1$, the integral

$$\int_a^b ds\, e^{iks}w(s) \approx w(a) \int_a^b ds\, e^{iks} \approx 0$$

Combining these two inequalities one can say that

$$\int_{-\infty}^{\infty} ds\, e^{iks}w(s) \approx 0$$

to the extent that k is large enough so that everywhere

$$\left|\frac{dw}{ds}\right| \frac{1}{k} \ll w$$

The actual calculation is straightforward. We want first to compute $Q(k)$ for small values of k. Expanding e^{iks} in Taylor's series, Eq. (1·10·7) becomes

$$Q(k) \equiv \int_{-\infty}^{\infty} ds\, w(s)\, e^{iks} = \int_{-\infty}^{\infty} ds\, w(s)(1 + iks - \tfrac{1}{2}k^2s^2 + \cdots)$$

or $\quad Q(k) = 1 + i\bar{s}k - \frac{1}{2}\overline{s^2}k^2 \cdots \qquad (1·11·1)$

where $\quad \overline{s^n} \equiv \int_{-\infty}^{\infty} ds\, w(s)s^n \qquad (1·11·2)$

is a constant which represents the usual definition of the nth moment of s. Here we assume that $|w(s)| \to 0$ rapidly enough as $|s| \to \infty$ so that these moments are finite. Hence (1·11·1) yields

$$\ln Q^N(k) = N \ln Q(k) = N \ln [1 + i\bar{s}k - \tfrac{1}{2}\overline{s^2}k^2 \cdots] \qquad (1·11·3)$$

Using the Taylor's series expansion valid for $y \ll 1$,

$$\ln(1+y) = y - \tfrac{1}{2}y^2 \cdots$$

$$\begin{aligned}\ln Q^N &= N[i\bar{s}k - \tfrac{1}{2}\overline{s^2}k^2 - \tfrac{1}{2}(i\bar{s}k)^2 \cdots] \\ &= N[i\bar{s}k - \tfrac{1}{2}(\overline{s^2} - \bar{s}^2)k^2 \cdots] \\ &= N[i\bar{s}k - \tfrac{1}{2}\overline{(\Delta s)^2}k^2 \cdots]\end{aligned}$$

where $\quad \overline{(\Delta s)^2} \equiv \overline{s^2} - \bar{s}^2$

Hence we obtain

$$Q^N(k) = e^{iN\bar{s}k - \frac{1}{2}N\overline{(\Delta s)^2}k^2}$$

Thus (1 · 10 · 8) becomes

$$\mathcal{P}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i(N\bar{s}-x)k - \frac{1}{2}N\overline{(\Delta s)^2}k^2}$$

$$\int_{-\infty}^{\infty} du\, e^{-au^2+bu} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

$$\mathcal{P}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$

$$\left.\begin{aligned}\mu &\equiv N\bar{s} \\ \sigma^2 &\equiv N\overline{(\Delta s)^2}\end{aligned}\right\}$$

We already showed that for the Gaussian distribution (1·6·4)

and $\quad \left.\begin{aligned}\bar{x} &= \mu \\ \overline{(\Delta x)^2} &= \sigma^2\end{aligned}\right\}$

Hence (1·11·9) implies

and $\quad \left.\begin{aligned}\bar{x} &= N\bar{s} \\ \overline{(\Delta x)^2} &= N\overline{(\Delta s)^2}\end{aligned}\right\}$